Indian Statistical Institute, Bangalore Centre
Solution set of M.Math II Year, End-Sem Examination 2015
Fourier Analysis

Note: We use the following notations $L^1(\mathbb{R}) = L^1$ and $L^2(\mathbb{R}) = L^2$.

1. Prove that $f, \hat{f} \in L^1$ then $f \in L^2$.

   Proof. Since $f \in L^1$, $\hat{f}$ is bounded by $\|f\|_{L^1}$. Then $\hat{f} \in L^2$ as $\hat{f} \in L^1$ which implies $f \in L^2$. □

2. Let $f(x) = \dfrac{x}{2}$ for $-\pi < x < \pi, 0$ for $x = \pm\pi$. Write down the Fourier series of f and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

   Proof. Since f is an odd function, its Fourier series coefficients $a_n = 0, n \geq 0$. Further
   $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx = \frac{(-1)^{n+1}}{n}, n \geq 1.$$
   Then Fourier series of f is the following
   $$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx.$$
   By Parseval's theorem we have
   $$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=1}^{\infty}\frac{1}{n^2},$$
   and it is easy to see that $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{\pi^2}{6}$. □

3. Let $f(x) = 1$ for $0 < x < \pi$, and $f(x) = -1$ for $-\pi < x < 0, 0, x = \pm\pi$. Find the discrete Hilbert transform of f.

   Proof. First note that the Fourier coefficients of the Fourier series of f are given by $a_n = 0, n \geq 0$ and $b_n = \dfrac{2(1-(-1)^n)}{n\pi}$. Then Fourier series of f is given by the
   $$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin nx = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\frac{e^{inx}-e^{-inx}}{2i}$$

The discrete Hilbert transform in the setting of the Fourier series is given by the formula

$$H\Big(\sum_{n\in\mathbb{Z}} c_n e^{inx}\Big) = -i\sum_{n\geq 1} c_n e^{inx} + i\sum_{n\leq -1} c_n e^{inx}.$$

Thus the discrete Hilbert transform is given by the

$$\sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n\pi}\frac{-ie^{inx}-ie^{-inx}}{2i} = \sum_{n=1}^{\infty} \frac{2(-1)^n)-1}{n\pi}\frac{e^{inx}+e^{-inx}}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^n)-1}{n\pi}\cos nx.$$

□

4. $f, g \in L^1, f, g \geq 0$ and $f = f * g$ then prove that $f = 0$ a.e..

Proof. Case 1: If $\hat{f}(0) = 0$, then

$$\|f\|_{L^1} = \int_{\mathbb{R}} |f(x)|dx = \int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x)e^{i0x}dx = \hat{f}(0) = 0$$

which implies $f = 0$ a.e..

Case 2: If $\hat{f}(0) \neq 0$, then there exists $\epsilon > 0$ such that $\hat{f}(t) \neq 0$ for $|t| < \epsilon$ as $\hat{f}$ is continuous. Since $f = f * g$ and there $\hat{f} = \hat{f}\hat{g}, \hat{g}(t) = 1$ for $|t| < \epsilon$. In particular $\hat{g}(t) = \hat{g}(0)$ for $|t| < \epsilon$ which implies

$$\int_{\mathbb{R}} g(x)e^{-itx}dx = \int_{\mathbb{R}} g(x)dx \Rightarrow \int_{\mathbb{R}} (1-\cos tx)g(x)dx = 0$$

which implies $g = 0$ a.e. as $g \geq 0$. Finally, $f = f * g = f * 0 = 0$ as an element of L^1. So $f = 0$ a.e.. □

5. Compute the Fourier transform of $x^2 e^{\frac{-x^2}{2}}$.

Proof. Since power series converge uniformly within all circles of convergence, and term-wise integration is valid for uniformly convergent series,

$$\begin{aligned}\hat{f}(s) &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ixs}\, dx \\ &= \int_{-\infty}^{\infty} \Big[\sum_{n=0}^{\infty} \frac{(-ixs)^n}{n!}\Big] e^{-\frac{x^2}{2}}\, dx\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-is)^n}{n!} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} x^n \, dx$$

The integral value is zero if n is odd; if $n = 2m$, then an application of Gamma function yields the following

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} x^{2m} \, dx = \sqrt{2\pi} \frac{(2m)!}{m! 2^m}.$$

Replacing these facts in the above expression it follows that

$$\hat{f}(s) = \sqrt{2\pi} e^{-\frac{s^2}{2}}.$$

Recall that $\widehat{x^n f(x)}(s) = i^n \frac{d^n}{ds^n} \hat{f}(s)$. Then for $n = 2$ we have

$$\widehat{x^2 e^{-\frac{x^2}{2}}}(s) = i^2 \frac{d^2}{ds^2} \sqrt{2\pi} e^{-\frac{s^2}{2}} = \sqrt{2\pi} e^{-\frac{s^2}{2}} (1 - s^2).$$

□

6. Show that if $f \in L^1$ and $\int_{\mathbb{R}} x^2 |\hat{f}(x)| dx < \infty$, then f is twice continuously differentiable.

Proof. First we will show that $\hat{f} \in L^1(\mathbb{R})$. Since $f \in L^1$, $\hat{f}$ is continuous and therefore it is bounded on $[-1, 1]$. Say it is bounded by $M > 0$.

$$\begin{aligned} \int_{\mathbb{R}} |\hat{f}(x)| dx &\leq \int_{|x| \leq 1} |\hat{f}(x)| dx + \leq \int_{|x|>1} |\hat{f}(x)| dx \\ 2M + &\leq \int_{|x|>1} x^2 |\hat{f}(x)| dx \\ &= 2M + \leq \int_{x \in \mathbb{R}} x^2 |\hat{f}(x)| dx < \infty \end{aligned}$$

Thus $\hat{f} \in L^1$. By the inverse Fourier transform we have $f(x) = \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt$. Now

$$\frac{\hat{f}(x+h) - \hat{f}(x)}{h} = \int_{\mathbb{R}} e^{itx} \frac{[e^{ith} - 1]}{h} \hat{f}(t) dt.$$

The integrand in the above equation is bounded by finite number for $|x| \leq 1$ and for $|x| > 1$ it is bounded by $x^2 |\hat{f}(x)|$ which is in L^1. Further, this integrand tends to $it\hat{f}(t)e^{itx}$ point-wise, hence by the Lebesgue dominated convergence theorem it converges to $it\hat{f}(t)e^{itx}$ in the L^1 norm. This implies that, as $h \to 0$, the right hand side of the above equation converges to inverse Fourier transform of $it\hat{f}(t)$ at x, i.e., $f'(x) = (it\hat{f})\check{}(x)$. By applying similar argument it can be shown that $\hat{f}$ is twice differentiable with $f''(x) = (i^2t^2\hat{f})\check{}(x)$ and the second derivative is continuous as it is inverse Fourier transform of $x^2\hat{f}(x) \in L^1$ function. $\square$

7. Let $f = I_{[0,1)}$. Prove that $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(t-2\pi n)|^2 = \frac{1}{2\pi}$ a.e. and hence prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{\sin^2(\pi t)}{(t+n)^2}$ is a constant. Compute this constant.

Proof. Since the integer translates $\{\phi(x-k) : k \in \mathbb{Z}\}$ of a scaling function forms an orthonormal family, i.e., for $k \neq m$

$$\begin{aligned}
0 &= \langle f(x-k), f(x-m)\rangle \\
&= \langle \hat{f}(x-k), \hat{f}(x-m)\rangle \qquad \text{(Parseval's Identity)} \\
&= \langle \hat{e}^{-itk}\hat{f}(t), e^{-itm}\hat{f}(t)\rangle \\
&= \int_{\mathbb{R}} e^{it(m-k)}|\hat{f}(t)|^2 dt.
\end{aligned}$$

Take $m - k = p$. Thus for $p \neq 0$ we have

$$\begin{aligned}
0 &= \int_{\mathbb{R}} e^{itp}|\hat{f}(t)|^2 dt \\
&= \sum_{n=-\infty}^{\infty} \int_{2\pi}^{2\pi(n+1)} e^{itp}|\hat{f}(t)|^2 dt \\
&= \sum_{n=-\infty}^{\infty} \int_{0}^{2\pi} e^{itp}|\hat{f}(t-2n\pi)|^2 dt \\
&= \int_{0}^{2\pi} e^{itp}[\sum_{n=-\infty}^{\infty} |\hat{f}(t-2n\pi)|^2] dt.
\end{aligned}$$

As a consequence of the above equations the 2π-periodic function

$$\sum_{n=-\infty}^{\infty} |\hat{f}(t-2n\pi)|^2$$

has Fourier coefficients c_n equal to zero for $n \neq 0$ and

$$c_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}.$$

Thus

$$\sum_{-\infty}^{\infty} |\hat{f}(t - 2n\pi)|^2 = \sum_{-\infty}^{\infty} c_n e^{int} = \frac{1}{2\pi}.$$

Since Fourier transform of the Haar scaling function $f = I_{[0,1)}$ is $\hat{f}(t) = e^{-it/2} \dfrac{\sin(t/2)}{t/2}$, the above formula for f gives us

$$\begin{aligned} \frac{1}{2\pi} &= \sum_{n=-\infty}^{\infty} |\hat{f}(t - 2n\pi)|^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{4\sin^2(t/2 - n\pi)}{(t - 2n\pi)^2} \quad \text{a.e.} \\ &= \sum_{n=-\infty}^{\infty} \frac{4\sin^2(t/2)}{(t - 2n\pi)^2} \quad \text{a.e.} \end{aligned}$$

Now replace t by $2\pi t$, we have

$$\sum_{n=-\infty}^{\infty} \frac{4\sin^2(t\pi)}{(2\pi t + 2\pi n)^2} = \frac{1}{2\pi} \quad \text{a.e.} \Rightarrow \sum_{n=-\infty}^{\infty} \frac{\sin^2(t\pi)}{(t+n)^2} = \frac{\pi}{2} \quad \text{a.e.} \qquad \square$$